Conceptualizing the core of the function concept – A facet model

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This paper presents an empirical model for conceptualizing the core of the function concept, which includes those facets that are equally important for all types of functions and common in all representations. The so-called facet model enables the identification of potential obstacles and a detailed description of students' learning processes when connecting representations (e.g. verbal and symbolic representations when solving word problems). In total 19 design experiments with overall 96 learners were conducted and qualitatively analyzed.

Keywords: Function understanding, concept formation, functions, lower secondary school.

Theoretical background

Conceptualizing function understanding

Functions are regarded as important learning content at school. Consequently, many theoretical or empirical studies have investigated the teaching and learning of functions since the beginning of the 20th century, working with very different conceptualizations of "function understanding" (Niss, 2014; for examples cf. Oehrtman, Carlson, & Thompson, 2008; Leinhardt, Zaslavsky, & Stein, 1990; Vinner & Dreyfus, 1989; common in German didactics is the concept of basic mental models (GVs), cf. vom Hofe, 2016). Niss emphasizes the complexity of the function concept and the necessity of "intentional and focused work on designing rich and multifaceted learning environments" (Niss, 2014, p. 240). Most studies share the overarching claim that students should really *understand* functions, even if this aim is interpreted in different ways, e.g. being able to identify a function (Vinner, & Dreyfus, 1989), having access to the process view of functions (Oehrtman et al., 2008) and further more. Accordingly, the conceptualizations differ and partly incorporate specific aspects of single representations or types of function. One example is the role of slope, which has specific importance for linear functions. This paper now presents a conceptualization of the core of the function concept as dependence relation, which is independent of (1) specific aspects of representations, (2) types of functions, and (3) different perspectives which still can and should all be taken. The facets building this core were empirically reconstructed and are summarized in Figure 1.

In (German) curricula the different function types (linear, square, exponential …) are introduced one after another. So when students deal with square functions it is not surprising that they tend to incongruously adopt specific attributes of linearity to square functions (e.g. Leinhardt et al., 1990). One reason might be that square functions are presented as „new" learning content without making explicit commonalities (especially the facets) and differences (e.g. the relevance and usability of constants for interpreting function equations). The aim of this study is to develop teaching-learning material, which makes this common facets explicit. Most approaches share the design principle *connecting representations* to initiate concept formation processes which is elaborated on below.

Connecting representations – learning medium and learning content

Studies dealing with connecting representations consider this activity in two different roles: On the one hand, connecting representations is regarded as useful medium for concept formation processes (cf. Duval, 2006). On the other hand, studies emphasize that connecting representations proves to be demanding for learners (Niss, 2014; Leinhardt et al., 1990). Therefore, connecting representations is not necessarily a resource one can build on when designing teaching-learning material, because it already requires some kind of conceptual understanding itself. So before using it as a resource for higher concept formation processes students' understanding of the core facets needs to be fostered. Many studies focus the symbolic, numerical and graphic representations (e.g. Moschkovich, 1998; Duval 2006). In this paper, the verbal representation will be used to explicate the common core facets. Dealing with functions means dealing with its representations (e.g. Duval, 2006; Leinhardt et al, 1990; Swan, 1985). However, the representations can be considered with different lenses or perspectives. Some perspectives are more obvious in one representation than in the other. This raises the question of the "core" of conceptual understanding of functions which is the same in all representations, as Niss mentions:

> One important issue that arises in this context is the fact that functions can be given several different representations (…), each of which captures certain, but usually not all, aspects of the concept. This may obscure the underlying commonality – the core – of the concept across its different representations, especially as translating from one representation to another may imply loss of information. (Niss, 2014, p. 240)

So considering connecting representations as a learning content raises the question of which detailed aspects building the core of the function concept students have to understand for being able to connect the representations. The core of the function concept shall include those aspects which are common in all representations and equal for all types of functions. This requires an adequate conceptualization to describe this core differentiated and explicitly.

Facet model

Conceptualizing the core of the function concept

The question of facets that build the core of the function concept is approached by using the construct of „comprehension elements" (Drollinger-Vetter, 2011), which is based on a cognitive psychological theory. Comprehension elements (further called: facets, indicated by ||…|| in the text) of a concept are defined as central mental schemes which are differently mirrored in different representations. By processes of unfolding and compacting processes of understanding are initiated. This construct is now adopted for the function concept. The facets have been reconstructed in the first design experiment cycle. When considering the common facets

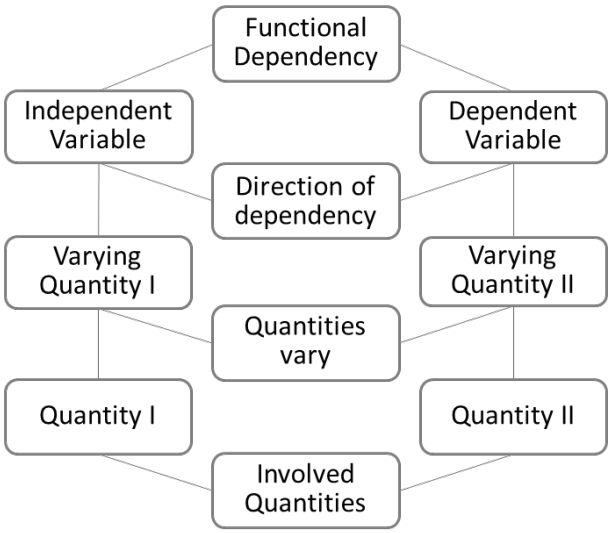

Figure 1 Facet model

for linear and square functions for example, one can identify that first it is important to know that there are two ||involved quantities||. General facets like this are shown in the middle, the concrete manifestations in the situation are shown in the outer paths. Having identified the concrete ||quantity I|| and ||quantity II|| students have to realize that these ||quantities vary|| and that the ||direction of dependency|| matters. These are the facets which are necessary to finally identify the two quantities as ||independent variable|| and ||dependent variable|| in the concrete situation. Considering the ||independent variable|| and ||dependent variable|| by describing the whole ||functional dependency|| is the most compacted way to talk about the core of the function concept. But when dealing with word problems or within learning processes it is equally important to be able to unfold compacted facets. Other "facets" as the ||meaning of the slope|| for example, are only helpful when dealing with linear functions. When dealing with square functions, the constants can only be interpreted in the graphic representation. Accordingly, using the facet model allows the following conceptualization of understanding the core of the function concept:

> "Conceptual understanding of [the core of] functional relationships can be defined as the ability to adopt different perspectives in different [representations] and to coordinate them by flexibly and adequately addressing the facets from [here: Figure 1]. The adequate addressing comprises flexible compacting and unfolding of conceptual facets, thus moving upwards and downwards in the facet model." (Prediger & Zindel, submitted, p.9)

This model has proven successful to identify and describe potential obstacles (for examples cf. Prediger & Zindel, submitted). Of course learners might address other or further facets than the normatively expected ones. The model is sensitive for these individual facets which can also be noticed and combined with other facets.

Research Questions

Connecting representations is not necessarily a resource that can be used to foster conceptual understanding, because it already requires some kind of conceptual understanding itself, namely flexibly unfolding and compacting the associated facets (Figure 1). This is a starting point to focus on the question of how to foster conceptual understanding. Teaching-learning material has been developed and empirically tested. In this paper the focus is on the following research question:

How can the facet model be used to describe and visualize learning processes (especially processes of connecting representations)?

Design

The methodology of this project is Topic-specific didactical Design Research (Prediger & Zwetzschler, 2013), which relies on an iterative interplay between designing teaching-learning material, conducting design experiments and analyzing the processes. In the overarching project, three design experiment cycles in laboratory setting and a fourth design experiment cycle in classroom setting were conducted. In total 39 learners participated in 16 design experiments in laboratory setting and further 57 learners participated in 3 design experiments in classroom setting (usually grade 9-10). The overall 42 sessions were videotaped (1890 minutes), partly transcribed and qualitatively analyzed.

Facet model as methodical framework to describe learning processes

Having introduced the facet model it can and should be used now as a starting point for fostering conceptual understanding by explicitly addressing its facets. The teaching-learning material intends to give the opportunity to get to know, address and combine facets from the facet model. Below, the therefor derived design principle of varying phrases is presented before an empirical insight in the effects is given.

Varying phrases – a design principle

Due to limitations in length of this paper Figure 2 shows only an excerpt of activities from the learning arrangement, realizing the design principles of connecting representations and systematic variation of phrases.

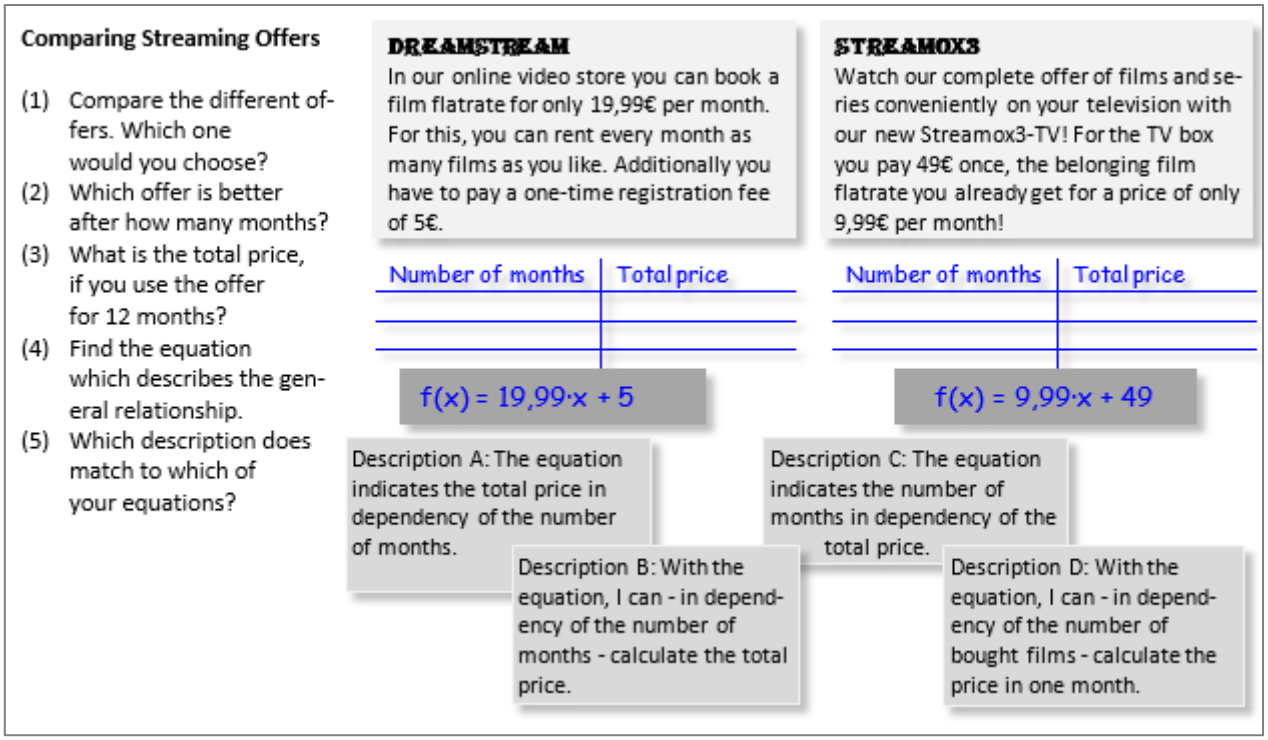

Figure 2: Excerpt from the learning arrangement (Descriptions A-D literally translated from German)

From a normative perspective different facets should be addressed by dealing with varied phrases. To achieve this all the phrases are varied in at least one of the facets. The sequence of considered phrases is not given.

Empirical insight: facet model as a starting point for fostering

The following short insight into the case study of Tatjana (15) and Alexandra (14) illustrates how dealing with varied phrases enables addressing and combining the facets from the model. To describe these processes the facet model is applied to visualize, which facets are addressed in each representation (here: verbal representations on the left, symbolic representation of the DreamStream offer on the right). Adequately addressed facets or connections are framed by green lines, inadequately addressed facets by red dashed lines. In order to connect representations it is necessary to address the same facets in both representations adequately.

100 Tatjana *[3s]* Well, the first one fits definitely *[points to "With the equation, I can - in dependency of the number of bought films - calculate the price in one month"]*.

101 Alexandra Yes, I think so, too. *[laughs]*

102 Tatjana *[laughs] [3s]* Because actually it doesn't matter how many films one takes. One still pays the same per month anyway.

First Tatjana (Figure 3) misjudges the matching of description D and explains it in 102 with the argument that it does not matter how many films one buys. She does not consider the two quantities in the phrase as ||varying quantities|| that are connected by a dependence relation. Instead she focuses on the two ||involved quantities|| and creates a connection between them herself. This connection does neither correspond to the phrase nor to the function equation. She identifies ||quantity I|| and ||quantity II|| in the equation, but she does not realize that these quantities are not the same as in the phrase. Following this, they pay attention to the next phrase.

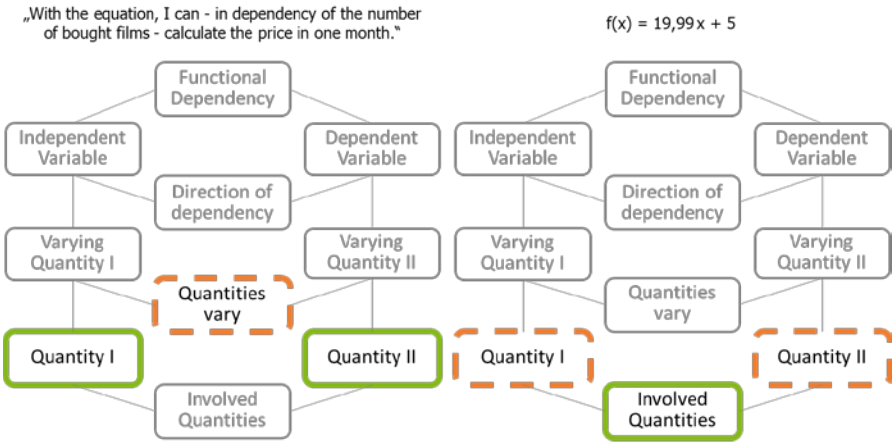

Figure 3 Tatjana (100 and 102)

104 Alexandra *[12s]* I think the second one is right, too *[points to "With the equation, I can - in dependency of the number of months - calculate the total price"]*. Because with the number of months, so this would be x indeed, hm calculate the total price, how much (…)

105 Tutor Mhm.

106 Tatjana *[11s]* This is the same like this *[points to "The equation indicates the total price in dependency of the number of months"]*, right?

107 Alexandra Yes.

Alexandra (Figure 4) correctly explains in 104 the matching of description B by identifying the same ||independent variable|| in the phrase and in the function equation. Moreover she addresses the ||functional dependency|| by ascertaining that one can calculate the total price (||dependent variable||) with the identified ||independent variable|| (number of months). Afterwards, Tatjana determines

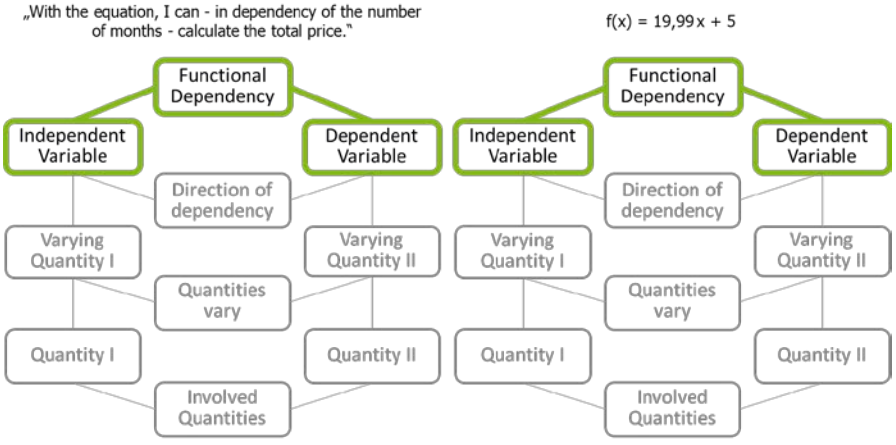

Figure 4 Alexandra (104)

that the descriptions B and A mean the same and only vary linguistically (106). When considering description C, the students struggle with connecting the representations.

109	Alexandra	*[25s]* With the fourth *[points to "The equation indicates the total price in dependency of the number of months"]* I don't know what is meant by it.
110	Tutor	What is meant by the phrase?
111	Alexandra	Yes, so what – I don't quite understand whether – so whether it is – put on the months or on the total price.
112	Tutor	I don't quite understand this.
113	Alexandra	Well here it says the number of months…
114	Tutor	Yes.
115	Alexandra	… is depending on the total price.
116	Tutor	Mhm.
117	Alexandra	But I don't quite know what is meant by it – well this – this phrase.

Alexandra (Figure 5) explicitly questions the ||independent variable|| in 111. She is able to identify the two ||varying quantities|| and she knows that the ||direction of dependency|| matters. But she cannot compact these facets to identify the ||independent variable|| in this situation. Finally in this scene the tutor asks Alexandra's opinion to description D.

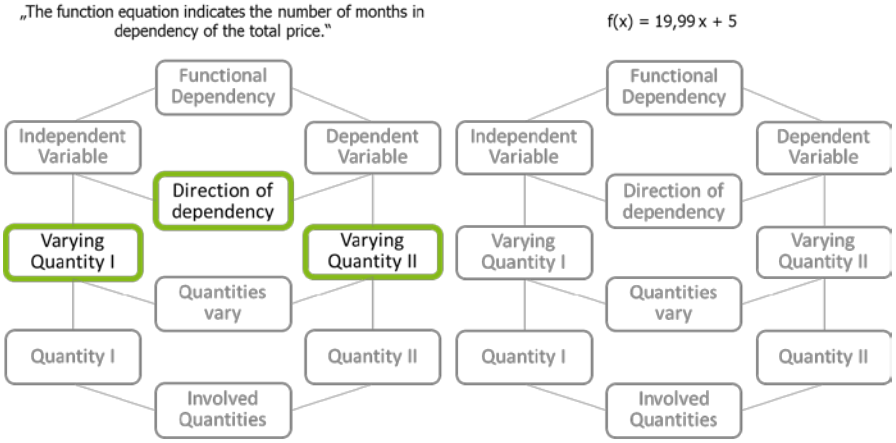

Figure 5 Alexandra (111)

141	Alexandra	Yes. I think so, too, that this is right, because one – x is indeed – are indeed the films and thereby one can just – noo these are the months! So I don't think that this is right. I think this is wrong.

In 141 Alexandra (Figure 6) reasserts her approval to Tatjana's judgment in 100 that the phrase fits to the DreamStream offer. She starts to explain this by reasoning about the meaning of the ||independent variable|| in the function equation and gets stuck. Beginning with interpreting the phrase she ascertains that the ||independent

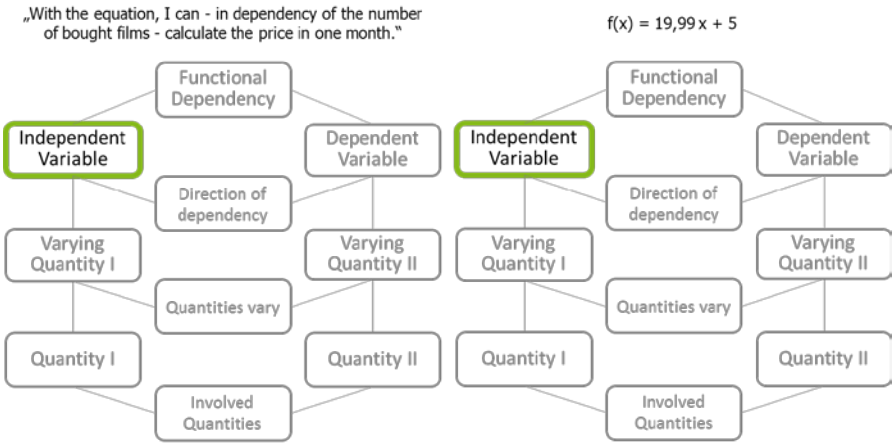

Figure 6 Alexandra (141) and Tatjana (146 and 148)

variable|| would be the number of bought films. But then she correctly states that this does not apply to the function equation because there the ||independent variable|| is the number of months. So she concludes that the phrase does not fit after all.

145	Tutor	*[laughing]* If you like to say something to this, here you are.
146	Tatjana	Yes, now I think, this is wrong.
147	Tutor	Why?
148	Tatjana	Well, because there it is put on the number of bought films.

Then Tatjana revises her first judgement, too. In 148 she explains it referring to the different ||independent variables||.

To sum up, contrasting the varied phrases initiated the addressing of different facets. The learning process is visible through increased precision and explicitness in students' utterances. Tatjana in particular focuses more and more on the meaning of the given phrase than on the situation itself.

Of course this excerpt is only an illustrating example of such a learning process. In other cases the developments look very different. One reason for this is that the sequence and number of considered phrases varied because it was not preset in the material but adopted for each process by the tutor. Overall, the empirical analysis of students' learning processes indicates the analytic power of the facet model and that dealing with varied phrases can foster addressing and combining facets.

Conclusion

This paper presented a conceptualization of function understanding focusing on the core of the function concept, which is based on cognitive psychological theories. It provides not only identifying potential obstacles but also a normative framework for fostering. Thereby the focus is on the facets which are necessary for understanding the core of the function concept. These facets are especially not specific for single representations or function types. Nevertheless it is of course important to learn specific knowledge about representations and different types of functions. But these facets build the core of function understanding which should be emphasized whenever students get to know new aspects of functions and which should be addressed consistently and again and again to make students aware of the core of functions. By dealing with varied phrases these facets can be addressed and processes of unfolding and compacting can be initiated. This has been illustrated in the empirical insight.

The facet model enables both, visualizing and better describing processes of connecting representations by contrasting the facets which are addressed in each representation. This enables not only describing the normatively prescribed facets but is also sensitive for individually activated facets. Because of the focus on processes it is not only diagnostic tool, but also usable to describe learning processes.

In the shown short empirical insight the model has been used to investigate the relation of verbal and symbolic representations when dealing with word problems. Presumably this is compatible for other relations of representations. Furthermore, the conceptualization presented here focuses on the over-

arching facets of the function concept. To what extent this can be combined with other conceptualizations to form a broader understanding of the function concept, ought to be subject of further analysis.

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